New properties of permutation groups

Marina Anagnostopoulou-Merkouri

Joint work with Rosemary Bailey, Peter Cameron, and Enoch Suleiman

24th October 2023

Part I

Pre-primitive groups

Primitivity and quasiprimitivity

$G \leqslant \mathrm{Sym}(\Omega)$ - transitive permutation group on a finite set Ω.

We say that $\Delta \subseteq \Omega$ is a **block** for G if

$$\Delta \cap \Delta^g \in \{\Delta, \emptyset\} \text{ for all } g \in G.$$

Note: $\Pi = \{\Delta^g \mid g \in G\}$ is a ***G*-invariant** partition of Ω.

We say that G is:

- **Primitive:** G is transitive and the only G-invariant partitions of Ω are the trivial ones.
- **Quasiprimitive:** all the non-trivial normal subgroups of G are transitive.

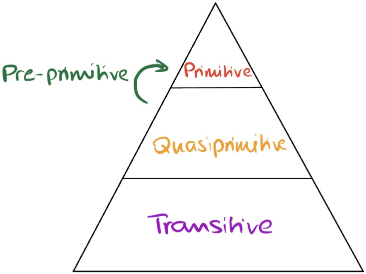
G primitive $\Rightarrow$ G quasiprimitive, G quasiprimitive $\not\Rightarrow$ G primitive.

"Lifting" quasiprimitivity to primitivity

Aim: Find a property P such that:

$$G \textbf{ quasiprimitive} + G \text{ has } P \iff G \textbf{ primitive.}$$

We will call this property **pre-primitivity**.



Definition. G is **pre-primitive (PP)** if every G-invariant partition is the orbit partition of a normal subgroup of G.

Lemma

Let $G \leqslant \mathrm{Sym}(\Omega)$. Then G is primitive if and only if it is both quasiprimitive and pre-primitive.

Proof. If G is primitive, then it is quasiprimitive and its G-invariant partitions are orbit partitions of G and 1 respectively.

Conversely, if G is pre-primitive, then each G-invariant partition is the orbit partition of some normal subgroup of G, so the only G-invariant partitions are the trivial ones.

An example and a non-example

Example

- $G = C_4 = \langle (1,2,3,4) \rangle$.
  - $\{1,2,3,4\} \longleftrightarrow G$;
  - $\{\{1,3\},\{2,4\}\} \longleftrightarrow \langle (1,3)(2,4) \rangle$;
  - $\{\{1\},\{2\},\{3\},\{3\}\} \longleftrightarrow 1$.
- $G = \langle (1,3,5)(2,4,6), (1,4)(2,3)(5,6) \rangle \cong S_3$.
  - $\{\{1,4\},\{2,5\},\{3,6\}\}$ is G-invariant;
  - The only non-trivial normal subgroup of G is $\langle (1,3,5)(2,4,6) \rangle$.

Motivation

Question: Can we classify all pre-primitive groups?

n	$T(n)$	$P(n)$	$PP(n)$	$QP(n)$	correlation
10	45	9	42	9	0.0133
11	8	8	8	8	0
12	301	6	276	7	0.0014
13	9	9	9	9	0
14	63	4	59	5	−0.0108
15	104	6	102	8	−0.0178
16	1954	22	1833	22	0.0007
17	10	10	10	10	0
18	983	4	900	4	0.0003
19	8	8	8	8	0
20	1117	4	1019	10	−0.0046

Observation: Pre-primitive groups are not "hard to find".

Regular groups

A transitive group $G \leqslant \mathrm{Sym}(\Omega)$ is **regular** if $G_\alpha = 1$ for any $\alpha \in \Omega$.

Note: If G is regular, then we can identify Ω with G and act by right multiplication.

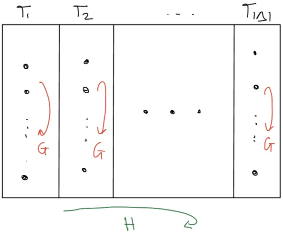
Theorem (A-M, Cameron, Suleiman, '23)

If $G \leqslant \mathrm{Sym}(\Omega)$ is regular, then it is PP if and only if it is a Dedekind group.

- A G-invariant partition is the right coset partition of some $H \leqslant G$.
- Conversely, every right coset partition is G-invariant.
- The left coset partition of H is the orbit partition of H.
- If $H \ntrianglelefteq G$, then the left coset partition of H and the right coset partition of H are different, and none of them have both properties.
- All subgroups of G must be normal.

The imprimitive wreath product

We first look at the imprimitive wreath product. Let $G \leqslant \mathrm{Sym}(\Gamma)$ and $H \leqslant \mathrm{Sym}(\Delta)$ and consider $G \wr H$.



Theorem (A-M, Cameron, Suleiman, '23)

The wreath product $G \wr H$ in its imprimitive action is pre-primitive if and only if both G and H are pre-primitive.

Some comments on the proof

G, H PP $\Rightarrow$ G $\wr$ H PP:

- Every $G \wr H$-invariant partition is comparable to the canonical partition in the $G \wr H$-invariant partition lattice.
- **Π - $G \wr H$-invariant partition above the canonical partition:** Canonical partition blocks are partitioned in some H-invariant way and this partition of the blocks is the orbit partition of some $N \trianglelefteq H$. Then we show that Π is the orbit partition of $G \wr N$.
- **Π - $G \wr H$-invariant partition below the canonical partition:** Each block of the canonical partition is partitioned in the same G-invariant way and this partition is the orbit partition of some $K \trianglelefteq G$. Then we show that Π is the orbit partition of $K^{|\Delta|}$.

The other direction is similar.

Direct products: A necessary condition

Unlike the wreath product case, it is not easy to find a necessary and sufficient condition for $G \times H$ in its product action to be pre-primitive.

Proposition (A-M, Cameron, Suleiman, '23)

If $G \times H$ is pre-primitive, then both G and H are pre-primitive.

- $G \times H$ can be embedded in $G \wr H$ in its imprimitive action.
- Since pre-primitivity is closed upwards, $G \wr H$ is pre-primitive.
- G and H are pre-primitive by the previous theorem.

Note: G, H PP $\not\Rightarrow G \times H$ PP. C_4 and Q_8 acting regularly are Dedekind and thus PP, but $C_4 \times Q_8$ is regular, but not Dedekind so not PP.

Direct products: Some facts about partitions

Let $G \leqslant \mathrm{Sym}(\Gamma)$ and $H \leqslant \mathrm{Sym}(\Delta)$. Every $(G \times H)$-invariant partition Π induces two partitions on Γ and Δ.

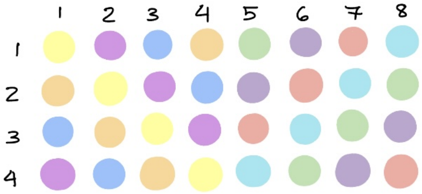
- Fix a $\delta \in \Delta$. The intersection of the parts of Π with $\Gamma \times \{\delta\}$ form a partition of $\Gamma \times \{\delta\}$, and by ignoring the second component we obtain a partition of Γ, which we call the ***G*-fibre partition** and we denote it by Π_G.
- The sets $\{\gamma \in \Gamma \mid (\exists \delta \in \Delta)(\gamma, \delta) \in P\}$ for every $P \in \Pi$ form a partition of Γ which we call the ***G*-projection partition** and we denote it by Π^G.

The H-fibre and H-projection partitions are defined in the same way.

Orbit and projection partitions: An example

Let $G = C_4 \leqslant S_4$ and
$H = Q_8 = \langle (1,2,3,4)(5,6,7,8), (1,5,3,7)(2,8,4,6) \rangle \leqslant S_8$.

The partition below is $(G \times H)$-invariant.



- **G-fibre partition:** Partition into singletons.
- **G-projection partition:** Partition into a single part.
- **H-fibre partition:** Partition into singletons.
- **H-projection partition:** $\{\{1,2,3,4\},\{5,6,7,8\}\}$.

Direct products: Some sufficient conditions

There are some special cases in which we know that $G \times H$ in its product action is pre-primitive.

Theorem (A-M, Cameron, Suleiman, '23)

In each of the following cases $G \times H$ is pre-primitive.

- G, H abelian;
- G, H are primitive;
- G, H are pre-primitive and $(|\Gamma|, |\Delta|) = 1$;
- G, H are pre-primitive and one of the following holds for every $(G \times H)$-invariant partition Π.
  - $\Pi_G = \Pi^G$ and $\Pi_H = \Pi^H$;
  - The Π^G and Π^H are the partitions into a single part.

Part II

Permutation groups and orthogonal block structures

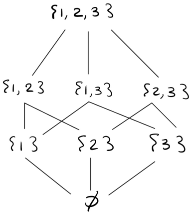
Lattices

A **lattice** L is a partially ordered set where for every $a, b \in L$ there exists a unique greatest lower bound (**meet**) $a \wedge b$, and a unique least upper bound (**join**) $a \vee b$.

We say that L is:

- **Modular:** if $a \leq b$ implies $a \vee (x \wedge b) = (a \vee x) \wedge b$ for all $a, b, x \in L$.
- **Distributive:** if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.

Example: If X is a set, then $(\mathcal{P}(X), \subseteq)$ is a lattice.



Orthogonal block structures

Let L be a lattice of partitions.

L is an **orthogonal block structure** if:

- All the partitions are uniform;
- For any $\Pi, \Sigma \in L$ the corresponding equivalence relations ρ_Π and ρ_Σ commute, i.e. $\rho_\Pi \circ \rho_\Sigma = \rho_\Sigma \circ \rho_\Pi$.

Motivation:

- Orthogonal block structures were first introduced by John Nelder in the area of experimental designs.
- Used when there are systematic differences between experimental units, e.g. patients in different hospitals.

OB groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be transitive.

Note: The G-invariant partitions form a lattice $L(G)$.

Definition. G is an **OB group** if $L(G)$ is an orthogonal block structure.

Theorem (A-M, Bailey, Cameron, '23+)

Let $\alpha \in \Omega$. Then G is OB if and only if for any two H, K such that $G_\alpha \leqslant H, K \leqslant G$ we have $HK = KH$.

Example

If G is transitive and abelian, then for any H, K such that $1 = G_\alpha \leqslant H, K \leqslant G$ we have $HK = KH$.

PP and OB groups

It turns out that the PP and the OB property are related.

Theorem (A-M, Bailey, Cameron, '23+)

If $G \leqslant \mathrm{Sym}(\Omega)$ is PP, then it is OB.

Proof sketch.

- Subgroups containing G_α correspond to G-invariant partitions.
- Two G-invariant partitions commute if and only if the corresponding subgroups commute.
- G-invariant partitions of PP groups are orbit partitions of normal subgroups and normal subgroups commute.

Question: Is G PP if and only if it is OB?

Counterexample: `TransitiveGroup(8, 14)`.

Poset block structures

- $P = \{p_1, \dots, p_n\}$ - finite poset. Associate a positive integer n_i to each p_i.
- Ω - Cartesian product of the sets $\{1, \dots, n_i\}$.
- $D \subseteq P$ is a **downset** if $p \in D$ implies $q \in D$ for all $q \leq p$.

For each downset of P we define the following equivalence relation R_D on Ω:

$$R_D((x_1, \dots, x_n), (y_1, \dots, y_n)) \iff (\forall p_i \notin D)(x_i = y_i).$$

- $\{R_D \mid D \subseteq P \text{ downset}\}$ forms an orthogonal block structure called a **poset block structure.**
- If the partition lattice of $G \leqslant \mathrm{Sym}(\Omega)$ forms a poset block structure, we say that G is a **PB group**.

OB and PB

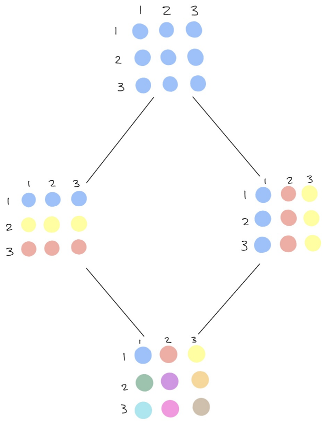
Lemma. An orthogonal block structure is a poset block structure if and only if it is distributive as a lattice, and hence an OB group is PB if and only if its partition lattice is distributive.

Remarks:

- Orthogonal block structures are modular as lattices.
- There are OB groups which are neither PB, nor PP, e.g. `TransitiveGroup(6, 2)` in the GAP Transitive Groups Library.
- There are OB groups which are PB, but not PP, e.g. `TransitiveGroup(8, 14)` in the GAP Transitive Groups Library.
- There are OB groups which are PP, but not PB, e.g. Q_8.

PB groups: An example

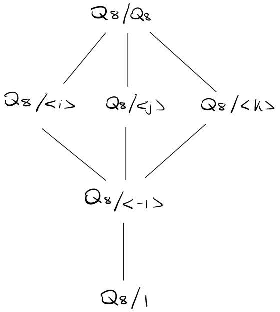
$G = S_3 \times S_3$



$L(G)$ is distributive, so G is PB.

PB groups: A counterexample

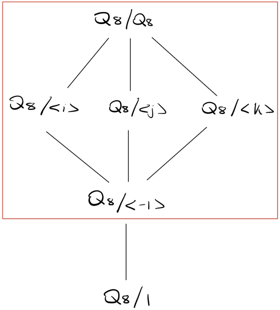
$$G = Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$$



A lattice is distributive if and only if it does not contain the diamond lattice and $L(Q_8)$ contains the diamond lattice, so Q_8 is not PB.

PB groups: A counterexample

$$G = Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$$



A lattice is distributive if and only if it does not contain the diamond lattice and $L(Q_8)$ contains the diamond lattice, so Q_8 is not PB.

Generalised wreath products I

Let $G \leqslant \mathrm{Sym}(\Gamma), H \leqslant \mathrm{Sym}(\Delta)$, and picture $\Gamma \times \Delta$ as a rectangular array.

- $G \times H$ in **product** action: permutation of the rows by an element of G followed by an **independent** permutation of the columns of $\Gamma \times \Delta$ by an element of H.
- $G \wr H$ in **imprimitive** action: independent permutation of the elements in each column by and element of G followed by a permutation of the columns by an element of H.

In the first case the actions of G and H on the array are independent, whereas in the second case the permutation of the columns by H **dominates** the action of G on each column.

Motivation: We might want to act on structures and induce different permutations of the structure that are either independent, or where some dominate others.

We can describe this domination relation in terms of a poset P of actions, where $p_1 \leq p_2$ if and only if p_2 dominates p_1.

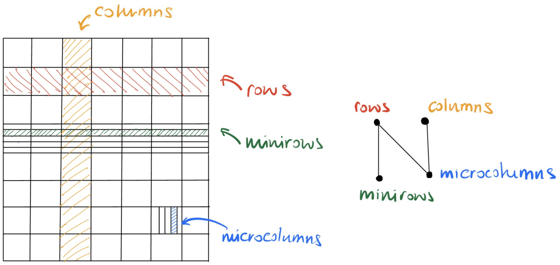


Figure: Permute rows, permute columns, within each row separately permute minirows, within each square separately permute microcolumns.

Generalised wreath products III

- Let P be the poset illustrated in the previous figure.
- Let G_1 permute microcolumns, G_2 permute the minirows, G_3 permute the rows, and G_4 permute the columns.
- We can define a group construction that acts on our structure in the way described in the caption, which we call the **generalised wreath product** of G_1, G_2, G_3, and G_4 over the poset P.
- We can do the same given any finite poset that describes domination of actions.

Remark: The direct product in its product action and the imprimitive wreath product are special cases of generalised wreath products.

Main theorem

Theorem (A-M, Bailey, Cameron, '23+)

Let I be a finite poset, and let G_i be a finite primitive group acting on a finite set Ω_i for every $i \in I$. Then the following hold:

1. The generalised wreath product G of the groups G_i over the poset I is pre-primitive, and hence OB;
2. The following are equivalent:
   - G has the PB property;
   - The only G-invariant partitions are the ones corresponding to downsets in I;
   - There do not exist incomparable elements $i, j \in I$ such that G_i and G_j are cyclic of the same prime order.

Some comments on the proof

- The direct product $\prod_{i \in I} G_i$ in its product action can be embedded transitively inside G.
- $\prod_{i \in I} G_i$ is pre-primitive.
- Since PP is closed upwards, G is pre-primitive.
- We prove part (ii) by considering possible partitions that do not correspond to downsets in I and deduce that we get extra ones if and only if there exist incomparable $i, j \in I$ such that $G_i \cong G_j \cong C_p$ for some prime p.

Embedding

Definition. Let L be a lattice of partitions. The **automorphism group** of L, denoted by $\mathrm{Aut}(L)$ is defined as the largest group preserving all partitions in L.

Theorem (Bailey, Praeger, Rowley, Speed, '82)

Let $P = \{p_1, \ldots, p_n\}$ be a finite poset and associate a positive integer n_i to each p_i. If $\mathcal{P}$ is the corresponding poset block structure, then $\mathrm{Aut}(\mathcal{P})$ is the generalised wreath product of the symmetric groups S_{n_i} over the poset P.

Corollary. Every PB group can be embedded in a generalised wreath product.

Current work. Can we do better than the full symmetric groups S_{n_i}?