The regularity number of a finite group

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Joint work with Tim Burness

Topics in Group Theory on the occasion of Andrea Lucchini's 60th(+) birthday



Bases of permutation groups

 Ω - finite set, $G \leq Sym(\Omega)$ transitive with point stabiliser H.

A base for G is a subset $\mathcal{B} \subseteq \Omega$ such that $G_{(\mathcal{B})} = \bigcap_{\beta \in \mathcal{B}} G_{\beta} = 1$.

The **base size** of *G*, denoted by $b(G, \Omega)$, is the minimal size of a base for *G*.

Note: $b(G, \Omega) \leq k \iff \exists g_1, \dots, g_k \in G$ such that

$$\bigcap_{i=1}^{k} H^{g_i} = 1$$

or equivalently G has a regular orbit on $(G/H)^k$.

The **base number** of G, denoted by B(G) is the maximum base size amongst all faithful transitive permutation reps of G.

Regular tuples

A tuple $\tau = (H_1, \ldots, H_k)$ of core-free subgroups of G is **regular** if there exist $g_1, \ldots, g_k \in G$ such that

 $\bigcap_{i=1}^{k} H_{i}^{g_{i}} = 1$

or equivalently if G has a regular orbit on

 $G/H_1 \times \cdots \times G/H_k$

We say that τ is **conjugate** if all its entries are conjugate.

Remark:
$$b(G, \Omega) \leq k \iff (\underbrace{H, \ldots, H}_{k})$$
 is regular.

Q: Smallest *k* s.t. all core-free *k*-tuples are regular?

The regularity number

The **regularity number** of G, denoted by R(G), is the smallest k such that all core-free k-tuples of G are regular.

Remark: B(G) is the smallest k such that all core-free **conjugate** k-tuples of G are regular.

If $S = \{H \leqslant G : H \text{ core-free}\}$ and $\mathcal{P} \subseteq S$, then we define:

• $R_{\mathcal{P}}(G) = \min\{k : \text{ every tuple in } \mathcal{P}^k \text{ is regular}\}$

• $B_{\mathcal{P}}(G) = \min\{k : \text{ every conj. tuple in } \mathcal{P}^k \text{ is regular}\}$

 $b(G, \Omega) \leqslant B(G) \leqslant R(G)$

Base conjectures

Some conjectures that have attracted a lot of attention in recent years are the following:

(1) **Cameron's conjecture:** If G is an almost simple group, then $B_{ns}(G) \leq 7$.

Burness et al. : Cameron's conjecture is true.

- (2) **Vdovin's conjecture:** $B_{sol}(G) \leq 5$ for every finite group.
 - Vdovin: reduction to almost simple groups
 - **Burness:** $B_{solmax}(G) \leq 5$ & sporadic socle
 - **Baykalov:** alternating socle & current work on classical groups

Generalised base conjectures

In view of the previous conjectures, we propose the following **generalised base conjectures**:

Conjecture 1: If G is almost simple, then $R_{ns}(G) \leq 7$ with equality if and only if $G = M_{24}$.

Conjecture 2: $R_{sol}(G) \leq 5$ for every finite group G.

Results

Theorem A (A-M & Burness 2024+)

Let G be almost simple with socle A_n . Then

• If
$$G \in \{S_n, A_n\}$$
, then $R(G) = n - |S_n : G|$

• $R_{ns}(G) \leq 6$, with $R_{ns}(G) = 2$ if $n \geq 13$

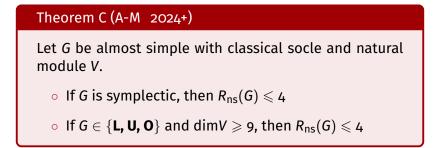
•
$$R_{solmax}(G) \leq 5$$
, with $R_{solmax}(G) = 2$ if $n \ge 17$

Theorem B (A-M & Burness 2024+)

Let G be almost simple with sporadic socle. Then

 $\circ R(G) \leqslant$ 7 with equality if and only if $G = M_{24}$

•
$$R_{sol}(G) \leq 3$$



Methods: probabilistic, computational, combinatorial.

Future goals:

- **1.** Prove Conjecture 1 for all almost simple groups of Lie type
- **2.** Prove that $R_{sol max}(G) \leq 5$ for all almost simple groups of Lie type
- 3. Prove Conjecture 2 for S_n and A_n