The regularity number of a finite group

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Bases of permutation groups

Ω - finite set, $G \leqslant \mathrm{Sym}(\Omega)$ transitive with point stabiliser H.

A **base** for G is a subset $\mathcal{B} \subseteq \Omega$ such that $G_{(\mathcal{B})} = \bigcap_{\beta \in \mathcal{B}} G_\beta = 1$.

The **base size** of G, denoted by $b(G, \Omega)$, is the minimal size of a base for G.

Example: $b(\mathrm{GL}(V), V \setminus \{0\}) = \dim(V)$

Note: In general, computing $b(G, \Omega)$ is **hard**.

Remark: $b(G, \Omega)$ is the smallest k for which there exist $g_1, \ldots, g_k \in G$ such that $\bigcap_{i=1}^{k} H^{g_i} = 1$ (i.e. G has a regular orbit on $(G/H)^k$).

The **base number** of G, denoted by $B(G)$, is the maximum base size over all transitive faithful permutation representations of G.

Motivating question

G has a reg orbit on $(G/H)^k \iff H$ has a reg orbit on $(G/H)^{k-1}$

Question: When does H have a regular orbit on $(G/K)^k$ for some $K \leqslant G$ core-free?

For example, if H is a stabiliser of a k-set in S_n, then what is its base size when it acts on partitions of $\{1, \ldots, n\}$?

Regular tuples

A tuple $\tau = (H_1, \ldots, H_k)$ of core-free subgroups of G is **regular** if there exist $g_1, \ldots, g_k \in G$ such that

$$\bigcap_{i=1}^{k} H_i^{g_i} = 1$$

or equivalently if G has a regular orbit on

$$G/H_1 \times \cdots \times G/H_k$$

We say that τ is **conjugate** if all its entries are conjugate.

Remark: $b(G, \Omega) \leqslant k \iff (\underbrace{H, \ldots, H}_{k})$ is regular.

The regularity number

The **regularity number** of G, denoted by $R(G)$, is the smallest k such that all core-free k-tuples of G are regular.

Remark: $B(G)$ is the smallest k such that all core-free **conjugate** k-tuples of G are regular.

If $\mathcal{S} = \{H \leqslant G \,:\, H \text{ core-free}\}$ and $\mathcal{P} \subseteq \mathcal{S}$, then we define:

- $R_{\mathcal{P}}(G) = \min\{k \,:\, \text{every tuple in } \mathcal{P}^k \text{ is regular}\}$
- $B_{\mathcal{P}}(G) = \min\{k \,:\, \text{every conj. tuple in } \mathcal{P}^k \text{ is regular}\}$

$$b(G, \Omega) \leqslant B(G) \leqslant R(G)$$

Question: Can we find G with $B(G) < R(G)$?

- $G = \mathrm{GL}_n(2)$, $n \geqslant 5$, $V = \mathbb{F}_2^n$.
- If $\Omega = V$, then $b(G, \Omega) = n$, and so $B(G) \geqslant n$.
- If $G \curvearrowright \Omega$ is not subspace, then $b(G, \Omega) \leqslant 4$ (Burness, 2007)
- If Ω is a set of k-spaces and $e_1, \ldots, e_n$ is a basis for V, then

$$\mathcal{B} = \{\langle e_i, e_{i+1}, \ldots, e_{k-i+1}\rangle \;:\; 1 \leqslant i \leqslant n\}$$

  is a base, so $B(G) = n$.
- H - stabiliser of a 1-space, K - stabiliser of a $(n-1)$-space.
- $(\underbrace{H, \ldots, H}_{n-1}, \underbrace{K, \ldots, K}_{n-2})$ is not regular, so $R(G) \geqslant 2(n-1)$
- $R(G) - B(G)$ can be arbitrarily large!

Base conjectures

Some conjectures that have attracted a lot of attention in recent years are the following:

(1) **Cameron's conjecture:** If G is an almost simple group, then $B_{\mathrm{ns}}(G) \leqslant 7$.

Burness et al. : Cameron's conjecture is true.

(2) **Vdovin's conjecture:** $B_{\mathrm{sol}}(G) \leqslant 5$ for every finite group.

- **Vdovin:** reduction to almost simple groups
- **Burness:** $R_{\mathrm{sol\,max}}(G) \leqslant 5$ & sporadic socle
- **Baykalov:** alternating socle & current work on classical groups

Generalised base conjectures

In view of the previous conjectures, we propose the following **generalised base conjectures**:

Conjecture 1: If G is almost simple, then $R_{\mathrm{ns}}(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$.

Conjecture 2: $R_{\mathrm{sol}}(G) \leqslant 5$ for every finite group G.

Results

Theorem A (A-M & Burness | 2024+)

Let G be almost simple with socle A_n. Then

- If $G \in \{S_n, A_n\}$, then $R(G) = n - |S_n : G|$
- $R_{\mathrm{ns}}(G) \leqslant 6$, with $R_{\mathrm{ns}}(G) = 2$ if $n \geqslant 13$
- $R_{\mathrm{sol\,max}}(G) \leqslant 5$, with $R_{\mathrm{sol\,max}}(G) = 2$ if $n \geqslant 17$

Theorem B (A-M & Burness | 2024+)

Let G be almost simple with sporadic socle. Then

- $R(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$
- $R_{\mathrm{sol}}(G) \leqslant 3$

Theorem C (A-M | 2024+)

Let G be almost simple with classical socle and natural module V.

- If $\dim V \geqslant 11$, then $R_{\mathrm{ns}}(G) \leqslant 4$
- If G is symplectic and $\dim V \geqslant 6$, then $R_{\mathrm{ns}}(G) \leqslant 4$

Methods: probabilistic, computational, combinatorial.

Future goals:

1. Prove Conjecture 1 for all almost simple groups of Lie type
2. Prove that $R_{\mathrm{sol\,max}}(G) \leqslant 5$ for all almost simple groups of Lie type
3. Prove Conjecture 2 for S_n and A_n