The regularity number of a finite group

Marina Anagnostopoulou-Merkouri

Joint work with Tim Burness

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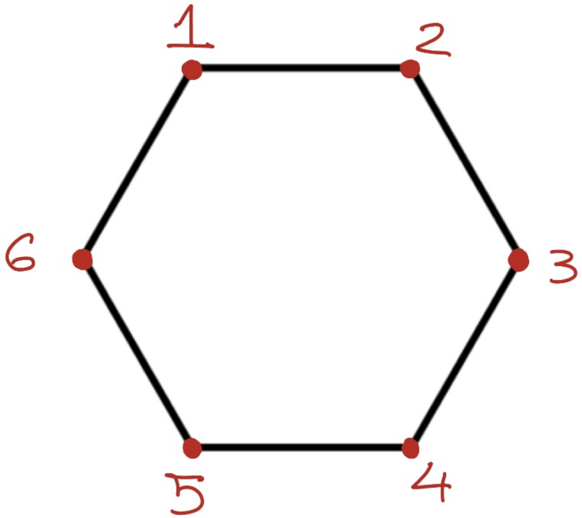
University of BRISTOL

Killing symmetries



Bases of permutation groups

Ω - finite set, $G \leqslant \mathrm{Sym}(\Omega)$ transitive, $\alpha \in \Omega$.

The **stabiliser** of α in G is

$$H = G_\alpha = \{g \in G \,:\, \alpha^g = \alpha\}$$

A **base** for G is a subset $\mathcal{B} \subseteq \Omega$ such that $\bigcap_{\beta \in \mathcal{B}} G_\beta = 1$.

The **base size** of G, denoted by $b(G, \Omega)$ is the smallest cardinality of a base for G.

Remark: $b(G, \Omega)$ is the smallest k for which there exist $g_1, \ldots, g_k \in G$ such that $\bigcap_{i=1}^{k} H^{g_i} = 1$.

Regular tuples

$H \leqslant G$ is **core-free** if $\bigcap_{g \in G} H^g = 1$.

A tuple $\tau = (H_1, \ldots, H_k)$ of core-free subgroups of G is **regular** if there exist $g_1, \ldots, g_k \in G$ such that

$$\bigcap_{i=1}^{k} H_i^{g_i} = 1$$

Remark: $b(G, \Omega) \leqslant k \iff \underbrace{(H, \ldots, H)}_{k}$ is regular.

What is the smallest k s.t. all c.f. k-tuples are regular?

The regularity number

The **regularity number** of G, denoted by $R(G)$, is the smallest k s.t. all core-free k-tuples of G are regular.

If $\mathcal{S} = \{H \leqslant G \,:\, H \text{ core-free}\}$ and $\mathcal{P} \subseteq \mathcal{S}$, then we define:

$$R_{\mathcal{P}}(G) = \min\{k \,:\, \text{every tuple in } \mathcal{P}^k \text{ is regular}\}$$

The almost simple groups

The Classification of Finite Simple Groups (CFSG)

If G is a finite simple group, then it belongs in one of the following classes:

- C_p (p prime)
- A_n ($n \geqslant 5$)
- Lie type (Classical or Exceptional)
- 26 sporadic simple groups

G is **almost simple** if there exists a nonabelian simple group T such that $T \trianglelefteq G \leqslant \mathrm{Aut}(T)$.

T is the **socle** of G.

Standard subgroups

$G \leqslant \mathrm{Sym}(\Omega)$ is **primitive** if it is transitive and its stabilisers are maximal subgroups.

Prim actions of G $\longleftrightarrow$ c.f. maximal subgps of G

G - almost simple:

- Standard actions $\longleftrightarrow$ large bases
- Non-standard actions $\longleftrightarrow$ small bases

Cameron's base size conjecture

Cameron's base size conjecture

If $G \leqslant \mathrm{Sym}(\Omega)$ is almost simple primitive with point stabiliser H and H is non-standard, then $b(G, \Omega) \leqslant 7$.

Burness et al. : Cameron's conjecture is true.

Generalised Cameron's conjecture

If G is almost simple, then $R_{ns}(G) \leqslant 7$.

Results

Theorem A (A-M & Burness | 2024+)

Let G be almost simple with socle A_n. Then

- If $G \in \{S_n, A_n\}$, then $R(G) = n - |S_n : G|$
- $R_{\mathrm{ns}}(G) \leqslant 6$, with $R_{\mathrm{ns}}(G) = 2$ if $n \geqslant 13$

Theorem B (A-M & Burness | 2024+)

If G is almost simple with sporadic socle, then $R(G) \leqslant 7$.

Theorem C (A-M | 2024+)

Let G be almost simple with classical socle and natural module V.

- If G is symplectic, then $R_{\mathrm{ns}}(G) \leqslant 4$
- If $G \in \{\mathbf{L}, \mathbf{U}, \mathbf{O}\}$ and $\dim V \geqslant 11$, then $R_{\mathrm{ns}}(G) \leqslant 4$

Methods: probabilistic, computational, combinatorial.

Other problems: $R_{\mathrm{sol}}(G)$, $R_{\mathrm{sol\,max}}(G)$, $R(G)$ for classical groups, algebraic groups.