The regularity number of a finite group

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Bases of permutation groups

Ω - finite set, $G \leqslant \mathrm{Sym}(\Omega)$.

A **base** for G is a subset $\mathcal{B} \subseteq \Omega$ such that $G_{(\mathcal{B})} = \bigcap_{\beta \in \mathcal{B}} G_\beta = 1$.

The **base size** of G, denoted by $b(G, \Omega)$, is the minimal size of a base for G.

Examples

- $b(S_n, \{1, \ldots, n\}) = n - 1$
- $b(\mathrm{GL}(V), V) = \dim(V)$

Note: In general, computing $b(G, \Omega)$ is **hard**.

Base size conjectures

Some conjectures that have attracted a lot of attention in recent years are the following:

(1) **Cameron's conjecture:** If G is an almost simple primitive group in a non-standard action, then $b(G, \Omega) \leqslant 7$.

Burness et al. : Cameron's conjecture is true.

(2) **Vdovin's conjecture:** If G is transitive with soluble point stabiliser, then $b(G, \Omega) \leqslant 5$.

- **Vdovin:** reduction to almost simple groups
- **Burness:** primitive groups & sporadic socle
- **Baykalov:** alternating socle & current work on classical groups

Regular tuples

Remark: If G is transitive with point stabiliser H, then $b(G,\Omega)$ is the smallest k such that G has a **regular orbit** on $(G/H)^k$.

A tuple $\tau = (H_1, \dots, H_k)$ of core-free subgroups of G is **regular** if G has a regular orbit on

$$G/H_1 \times \cdots \times G/H_k$$

or equivalently if there exist $g_1, \dots, g_k \in G$ such that

$$\bigcap_{i=1}^{k} H_i^{g_i} = 1$$

Remark: If G is transitive with point stabiliser H, then $b(G,\Omega) \leqslant k \iff$ the k-tuple $(H, \dots, H)$ is regular.

The regularity number

The **regularity number** of G, denoted by $R(G)$, is the smallest k such that all core-free k-tuples of G are regular.

If $\mathcal{S} = \{H \leqslant G \,:\, H \text{ core-free}\}$ and $\mathcal{P} \subseteq \mathcal{S}$, then we define:

$$R_{\mathcal{P}}(G) = \min\{k \,:\, \text{every tuple in } \mathcal{P}^k \text{ is regular}\}$$

We propose the following **generalised base conjectures**:

- **Conjecture 1:** If G is almost simple, then $R_{\mathrm{ns}}(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$.
- **Conjecture 2:** We have $R_{\mathrm{sol}}(G) \leqslant 5$ for every finite group.

Results

Theorem A (A-M & Burness | 2024+)

Let G be almost simple with socle A_n. Then

- If $G \in \{S_n, A_n\}$, then $R(G) = n - |S_n : G|$
- $R_{\mathrm{ns}}(G) \leqslant 6$, with $R_{\mathrm{ns}}(G) = 2$ if $n \geqslant 13$
- $R_{\mathrm{sol\,max}}(G) \leqslant 5$, with $R_{\mathrm{sol\,max}}(G) = 2$ if $n \geqslant 17$

Theorem B (A-M & Burness | 2024+)

Let G be almost simple with sporadic socle. Then

- $R(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$
- $R_{\mathrm{sol}}(G) \leqslant 3$

Final remarks

Methods: probabilistic, computational, combinatorial.

Future goals:

1. Prove Conjecture 1 for all almost simple groups of Lie type
2. Prove that $R_{\mathrm{sol\,max}}(G) \leqslant 5$ for all almost simple groups of Lie type
3. Prove Conjecture 2 for S_n and A_n